Problems with Multiple Objectives (Multicriteria Optimization)

$$Min \quad f_1(\mathbf{x}) \qquad Min \quad f_2(\mathbf{x}) \qquad Max \quad f_3(\mathbf{x}) \qquad Min \quad f_M(\mathbf{x})$$

$$F(\mathbf{x}) = f[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_M(\mathbf{x})]$$

- Individual objectives are usually in contradiction with one another, hence
- If $\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_M^*$ are the solutions to individual objectives, then $\mathbf{x}_1^* \neq \mathbf{x}_2^* \neq \ldots \neq \mathbf{x}_M^*$

- If the individual objectives are controlled by different sets of variables:

$$f_1(\boldsymbol{x}) = f_1(\boldsymbol{x}_1) \qquad f_2(\boldsymbol{x}) = f_2(\boldsymbol{x}_2) \qquad f_M(\boldsymbol{x}) = f_M(\boldsymbol{x}_M)$$

separable function

$$F(\boldsymbol{x}) = f_1(\boldsymbol{x}_1) + f_2(\boldsymbol{x}_2) + \ldots + f_M(\boldsymbol{x}_M) = \sum_{i}^{M} f_i(\boldsymbol{x}_i)$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M)$

$$\boldsymbol{x}_1 = (x_1, \ldots, x_t) \qquad \boldsymbol{x}_2 = (x_{t+1}, \ldots, x_s) \qquad \boldsymbol{x}_M = (x_{q+1}, \ldots, x_n)$$

- Then, the optimum of f can be obtained by optimizing the individual f_i’s.

$$Min \;\; f_1(\boldsymbol{x}) \qquad Min \;\; f_2(\boldsymbol{x}) \qquad Max \;\; f_3(\boldsymbol{x}) \qquad Min \;\; f_M(\boldsymbol{x})$$

$$\boldsymbol{x}_1^*, f_1^* \qquad \boldsymbol{x}_2^*, f_2^* \qquad \boldsymbol{x}_3^*, f_3^* \qquad \boldsymbol{x}_M^*, f_M^*$$

$$\boldsymbol{x}^* = (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_M^*)$$

- All objectives are controlled by the same set of variables:

(a) Composite objective function:

$$f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \ldots + \alpha_M f_M(\mathbf{x})$$

(b) Choose the most important to Max (Min), and put limits on the others.

Optimize $f(\mathbf{x}) = f_2(\mathbf{x})$

such that $f_1(\mathbf{x}) \geq A_1 \quad f_3(\mathbf{x}) \geq A_3 \quad \ldots \quad f_M(\mathbf{x}) \geq A_M$

(c) Optimize each of the objectives with respect to $\boldsymbol{x}$ individually to find f_i^* and the corresponding $\boldsymbol{x}_i^*$

$$\textbf{Min or Max} \quad f_i \quad f_i^*, \boldsymbol{x}_i^* \qquad \textbf{where} \quad i = 1, 2, \ldots\ldots, M$$

$$\boldsymbol{x}_1^* \neq \boldsymbol{x}_2^* \neq \ldots \neq \boldsymbol{x}_M^* \qquad \textbf{each design is distinct}$$

Then,

$$\textbf{either Min} \; \underset{i=1,..,M}{Max}[d_i(\boldsymbol{x})] \qquad \textbf{or Min} \sum_{i=1}^{M} d_i^2 \qquad \textbf{where} \quad d_i(\boldsymbol{x}) = \frac{f_i(\boldsymbol{x}) - f_i^*}{f_i^*}$$

- A vector of design variables $\boldsymbol{x}^*$ is said to be Edgeworth - Pareto optimal if for any other design variable vector $\boldsymbol{x}$ the values of the objective functions f_i either remain the same or at least one of them worsens.